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SPECTRAL EQUIVALENCE OF DETERMINISTIC AND STOCHASTIC MODELS FOR--ETC(U)
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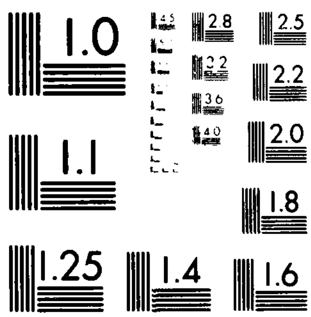

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May 1982

SPECTRAL EQUIVALENCE OF DETERMINISTIC AND STOCHASTIC MODELS FOR TIME SERIES WITH A PERIODIC COMPONENT

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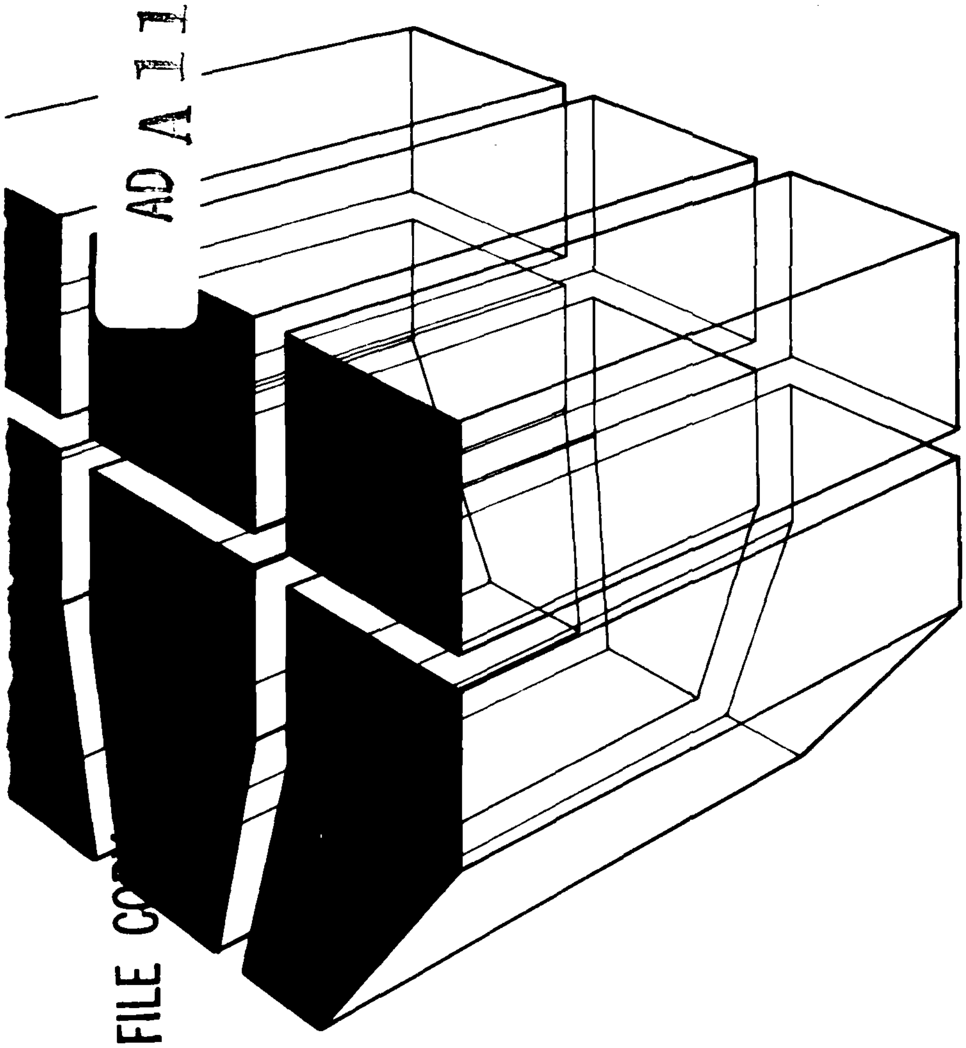

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Time series encountered in engineering applications typically contain strongly periodic behavior because of the dynamics of the underlying physical system. Several different approaches are being used to model this type of time series. However, in the time series literature there is little agreement about the appropriateness of alternative models.

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Block 20 continued.

A fundamental knowledge of the similarities and differences among alternative models is gained by understanding how alternative models account for the variance in periodic time series.

This report presents the concept that a deterministic and stochastic model are equivalent when their spectra represent the same frequency decomposition of the variance. The graphical interpretation of the poles and zeros of the associated polynomial equations in the unit circle on the complex plane and the analytic expression for the summation from zero to infinity of the square of the coefficients of Green's function are used to identify an equivalent stochastic model for a given deterministic model.

Thus spectral equivalence provides a criterion for evaluating the appropriateness of alternative models for time series with a periodic component.

FOREWORD

This research was conducted under Project 4A161101A91D, "In-Laboratory Independent Research Program (ILIR)"; Scientific Area 04; Work Unit 097, "Decomposition of Periodic Time Series." This work was performed by the Facility Systems Division (FS) of the U.S. Army Construction Engineering Research Laboratory (CERL). Mr. E. A. Lotz is Chief of CERL-FS.

This research was conducted in partial fulfillment of the requirements for the degree of Master of Science in Industrial Engineering at the University of Illinois at Urbana-Champaign.

I wish to express my appreciation to Professor R. E. DeVor and Professor S. G. Kapoor, my advisors, for their guidance during the course of this research and their helpful comments on this manuscript. I am also grateful to Dr. D. C. Hittle and Professor G. J. Janusz for the many discussions we had during this research.

COL Louis J. Circeo is Commander and Director of CERL, and Dr. L. R. Shaffer is Technical Director.

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1. INTRODUCTION

Many time series encountered in engineering applications are strongly periodic -- for example, outdoor temperature, electrical consumption, the roughness of machined surfaces, and machine tool chatter vibration. This periodic behavior is caused by the dynamics of the underlying physical system. Engineers are interested in analyzing these time series in order to characterize, forecast, and control the underlying system. There are several different approaches currently being used to model this type of time series. However, there is little agreement in the time series literature on the appropriateness of alternative models.

The first approach, developed by Box and Jenkins (1970), advocates the use of a seasonal difference operator and seasonal autoregressive-moving average parameters. These are then combined with non-seasonal differencing and non-seasonal autoregressive-moving average parameters, as appropriate, resulting in a multiplicative autoregressive-integrated-moving average (ARIMA) model. This approach was used by Jenkins (1979) to model outdoor temperature and electrical consumption, by Kline (1979) to model outdoor temperature and the roughness of machined surfaces, and by DeVor and Wu (1971) to model surface roughness.

The second approach, known as DDS -- developed by Pandit and Wu, Pandit (1973), Wu (1977), and Pandit and Wu (1982) -- advocates the use of models of the form ARMA(n,n-1). This approach was used by Kline

(1979) to model outdoor temperature and surface roughness, by O'Connor and Kapoor (1982) to model electrical consumption, and by DeVries (1979) to model tool chatter vibration in a single point turning operation. This approach generally resulted in the identification of high order ARMA models. For example, DeVries found an ARMA(10,9) for a chatter vibration model, and Kline found an ARMA(8,6) was required to adequately model temperature.

The third approach consists of modeling periodic time series with a combined deterministic plus stochastic model. Typically, the deterministic portion is modeled by a sum of relatively few sinusoids and the stochastic portion by a relatively low order ARMA model. This approach essentially extends the classical curve fitting approach which assumes that independently and identically distributed errors (white noise) are superpositioned on deterministic functions. Anderson (1971) discusses the problems of statistical inference both under the classical error assumptions and when the random element satisfies a stochastic difference equation. This approach was used by Kline (1979) to model temperature and surface roughness; by Hittle (1981), Hittle and Pedersen (1981) to model temperature; and by O'Connor and Kapoor to model electrical consumption.

Kline studied the relative merit, based on the one-step and ℓ-step ahead forecast error, of Box-Jenkins, DDS, and deterministic plus stochastic models for several periodic time series. In general, he found that deterministic plus stochastic models provided better forecasts than either the Box-Jenkins or DDS models. O'Connor and Kapoor found that a

deterministic plus stochastic model provided better forecasts than the models identified by DDS for electrical consumption for all forecasts from one to 24 hours ahead. A similar result was reported by Pandit and Wu in modeling the classical airline data originally presented by Box and Jenkins. Box and Jenkins concluded that an ARIMA model was adequate for forecasting the airline data.

DeVries used the DDS approach to model tool chatter in order to characterize the vibration dynamics for control purposes. He speculates that the relatively high order ARMA models identified by DDS may be able to be simplified via pole-zero cancellation. DeVries further notes that this would be particularly attractive, for simplified on-line control of turning operations, if an AR(2) were found to be sufficient.

Hittle and Pedersen developed ARMA models via the DDS approach and deterministic plus stochastic models to forecast outdoor temperature for use in the calculation of heat conduction through multi-layered building walls and floors. They note that this calculation is simplified if the outside surface temperature can be treated as the sum of pure sinusoids. Hittle and Pedersen conclude that both Fourier analysis and the DDS modeling procedure allow identification of periodic behavior. However, they further conclude, adequate ARMA models do not provide good estimates of the amplitude of the principal sinusoidal behavior.

The confusion and ambiguity in today's time series literature regarding the appropriateness of alternative models makes it difficult to select a valid modeling approach. Criteria are needed for selecting

appropriate modeling approaches for this type of strongly periodic time series. This research represents a first step towards establishing selection criteria. A fundamental understanding of the similarities and differences among alternative models will be gained by understanding the manner in which alternative models account for the variance in periodic time series. In particular, the objective of this thesis is to identify mathematically the spectral equivalency, when it exists, between a deterministic and a stochastic representation of a periodic component in a time series. Since the spectrum represents the frequency decomposition of the variance of the time series, this provides a criterion for selecting an appropriate model.

In order to establish this spectral equivalency, the spectrum for a discrete deterministic time series, consisting of a sinusoid plus white noise, is derived in Chapter 2. The spectrum for a second order Autoregressive-second order Moving Average, ARMA(2,2), stochastic model is derived in Chapter 3. Equivalency between deterministic and stochastic models is defined in Chapter 4 based on the frequency decomposition of the variance of the time series. A procedure for identifying an equivalent stochastic model for a given deterministic model is also presented. The identification procedure is illustrated with a numerical example in Chapter 5.

2. DETERMINISTIC SPECTRUM

Spectrum of a Sinusoid

Consider a periodic time series $\{x_t\}_{t=1}^{N}$ that consists of a single sinusoid plus white noise.

$$x_t = d_t + a_t \tag{Eq. 2-1}$$

where

$$d_t = A \cos (\omega t + \omega_o)$$

and

a_t is an independently and identically distributed random variable with mean 0 and variance σ_a^2; i.e.,

a_t is iid $(0,\sigma_a^2)$

and

a_t and d_t are independent.

The spectral density function, or more commonly the spectrum of $\{x_t\}$, will be denoted by $f_x(\theta)$ and is defined by the discrete Fourier transform, Jenkins & Watts (1968):

$$f_x(\theta) = \frac{1}{\pi N} \left| \sum_{t=1}^{N} x_t \, e^{i\theta t} \right|^2 \qquad 0 \leq \theta \leq \pi \qquad \text{(Eq. 2-2)}$$

Substituting $x_t = d_t + a_t$ yields

$$f_x(\theta) = \frac{1}{\pi N} \left| \sum_{t=1}^{N} d_t \, e^{i\theta t} + \sum_{t=1}^{N} a_t \, e^{i\theta t} \right|^2 \qquad \text{(Eq. 2-3)}$$

Since the complex absolute value squared of a complex number is the product of the complex number and its conjugate, Churchill (1974), expansion yields

$$f_x(\theta) = \frac{1}{\pi N} (\Sigma \, d_t e^{i\theta t} \Sigma d_t e^{-i\theta t} + \Sigma d_t e^{i\theta t} \, \Sigma a_t e^{-i\theta t}$$

$$+ \Sigma a_t e^{i\theta t} \, \Sigma d_t e^{-i\theta t} + \Sigma a_t e^{i\theta t} \, \Sigma a_t e^{-i\theta t}) \qquad \text{(Eq. 2-4)}$$

But

$$\sum_{t=1}^{N} d_t e^{i\theta t} \sum_{k=1}^{N} a_k e^{-i\theta k} = \sum_t \sum_k d_t a_k e^{i\theta(t-k)}$$

$$= \sum_{\ell} \left(\sum_t d_t \; a_{t-\ell}\right) e^{i\theta\ell}$$

where $\ell = t-k$

$$= 0 \qquad \text{(Eq. 2-5)}$$

Since $E(d_t a_t) = E(d_t) \cdot E(a_t) = 0$ which implies $\Sigma d_t a_t = 0$

Similarly

$$\Sigma a_t e^{i\theta t} \, \Sigma d_t e^{-i\theta t} = 0 \qquad \text{(Eq. 2-6)}$$

Hence Eq. 2-4 can be written as

$$f_x(\theta) = \frac{1}{\pi N} \left| \sum_{t=1}^{N} d_t e^{i\theta t} \right|^2 + \frac{1}{\pi N} \left| \sum_{t=1}^{N} a_t e^{i\theta t} \right|^2 \qquad \text{(Eq. 2-7)}$$

That is, the spectrum of the sum is the sum of the spectra when series are independent.

The spectrum of $\{d_t\}$ can be shown (Appendix A) to be

$$f_d(\theta) = \frac{A^2}{4\pi N}\left[\frac{\sin^2 \frac{N(\omega+\theta)}{2}}{\sin^2 \frac{\omega+\theta}{2}} + 2\cos[(N+1)\omega+2\omega_o]\frac{\sin\frac{N(\omega+\theta)}{2}}{\sin\frac{\omega+\theta}{2}} \cdot \frac{\sin\frac{N(\omega-\theta)}{2}}{\sin\frac{\omega-\theta}{2}} + \frac{\sin^2 \frac{N(\omega-\theta)}{2}}{\sin^2 \frac{\omega-\theta}{2}}\right] \qquad \text{(Eq. 2-8)}$$

Since the spectrum for a finite discrete series is only defined at discrete points, it is understood that $\theta = \frac{2\pi j}{N}$ $j = 0,1,2,\ldots,\frac{N}{2}$.

The spectrum can be evaluated at $\theta = \omega$ by applying ℓ'Hopital's rule to evaluate the indeterminate factor as follows:

$$\lim_{\theta\to\omega} \frac{\sin\frac{N(\omega-\theta)}{2}}{\sin\frac{(\omega-\theta)}{2}} = \lim_{\theta\to\omega} \frac{\dfrac{N\sin\frac{N(\omega-\theta)}{2}}{N(\frac{\omega-\theta}{2})}}{\dfrac{\sin\frac{\omega-\theta}{2}}{\frac{\omega-\theta}{2}}}$$

$$= N \qquad \text{(Eq. 2-9)}$$

Therefore, from Eq. 2-8 and Eq. 2-9, we get

$$f_d(\omega) = \frac{A^2}{4\pi N}\left[\frac{\sin^2 N\omega}{\sin^2\omega} + 2\cos[(N+1)\omega + 2\omega_o]\frac{N\sin N\omega}{\sin\omega} + N^2\right] \qquad \text{(Eq. 2-10)}$$

If N is chosen such that it is an integer multiple of the periodicity, i.e., if $\omega = \frac{2\pi k}{N}$ where k is a non-zero integer, then $\sin N\omega = 0$ and Eq. 2-10 reduces to

$$f_d(\omega) = \frac{A^2}{2} \frac{N}{2\pi} \qquad \text{(Eq. 2-11)}$$

For $\theta \neq \omega$ i.e. $\theta = \frac{2\pi j}{N}$, $\omega = \frac{2\pi k}{N}$ and $j \neq k$

$$\sin \frac{N(\omega+\theta)}{2} = \sin(k+j)\pi = 0 \qquad \text{(Eq. 2-12)}$$

and

$$\sin \frac{N(\omega-\theta)}{2} = \sin(k-j)\pi = 0 \qquad \text{(Eq. 2-13)}$$

Therefore, in this case Eq. 2-8 reduces to

$$f_d(\theta) = 0 \qquad \theta \neq \omega \qquad \text{(Eq. 2-14)}$$

Spectrum of White Noise

The spectrum of $\{a_t\}$ can be shown (Appendix B) to be

$$f_a(\theta) = \frac{\sigma_a^2}{\pi} \qquad \text{(Eq. 2-15)}$$

<u>Spectrum of a Deterministic Model</u>

Thus the spectrum of $\{x_t\}$, denoted by $f_x(\theta)$, where

$$x_t = A\cos\left(\frac{2\pi k}{N}t + \omega_o\right) + a_t \quad \text{is} \qquad \text{(Eq. 2-16)}$$

$$f_x\left(\frac{2\pi j}{N}\right) = \begin{cases} \dfrac{A^2}{2} \cdot \dfrac{N}{2\pi} + \dfrac{\sigma_a^2}{\pi} & j = k \\ \\ \dfrac{\sigma_a^2}{\pi} & j \neq k \end{cases} \qquad \text{(Eq. 2-17)}$$

$$\text{for } j = 0, 1, \ldots, \frac{N}{2}$$

This spectrum is shown graphically in Figure 2-1.

The spectrum represents the frequency decomposition of the variance of $\{x_t\}$. This can be seen by integrating the spectrum via the rectangular rule to show that the area is equal to the variance of $\{x_t\}$.

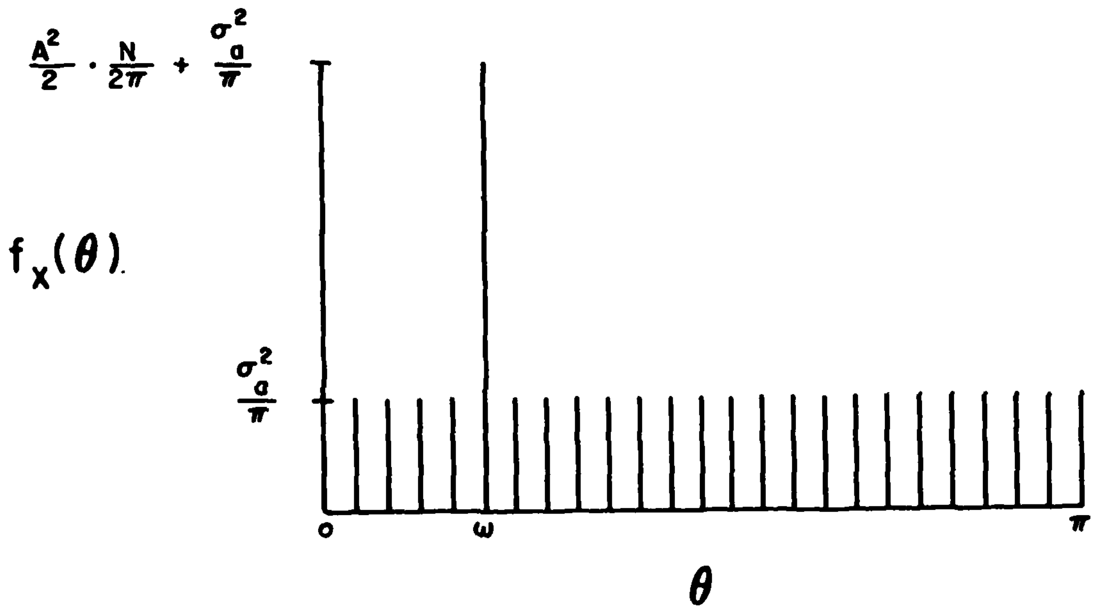

Figure 2-1. Spectrum of A $\cos(\omega t + \omega_o) + a_t$

$$\mathrm{Var}(x_t) = \frac{A^2}{2} \cdot \frac{N}{2\pi} \cdot \frac{2\pi}{N} + \frac{\sigma_a^2}{\pi} \cdot \frac{2\pi}{N} \cdot \frac{N}{2}$$

$$= \frac{A^2}{2} + \sigma_a^2 \qquad \text{(Eq. 2-18)}$$

That is, the variance of x_t is the sum of the variance of d_t and a_t since d_t and a_t are independent.

Note that Eq. 2-17 can be normalized by dividing by $\frac{\sigma_a^2}{\pi}$ (iff $\sigma_a^2 \neq 0$.)

$$\frac{\pi}{\sigma_a^2} f_x\left(\frac{2\pi j}{N}\right) \begin{cases} \frac{A^2}{2} \cdot \frac{N}{2\sigma_a^2} + 1 & j = k \\ 1 & j \neq k \end{cases} \qquad \text{(Eq. 2-19)}$$

for $j = 0, 1, \ldots, \frac{N}{2}$

In summary, we have shown in this chapter that the spectrum for a sinusoid plus white noise is equal to the sum of the spectrum for a sinusoid and the spectrum for white noise. The spectrum for a discrete sinusoid plus white noise will be used in Chapter 4 to establish equivalency between deterministic and stochastic models.

3. STOCHASTIC SPECTRUM

Stochastic Model

We now consider stochastic models of the form

$$y_t = \frac{\theta(B)}{\phi(B)} a_t = \psi(B)a_t \qquad \text{(Eq. 3-1)}$$

where $\theta(B)$ and $\phi(B)$ represent polynomials in B, the back shift operator, which is defined such that $Bx_t = x_{t-1}$, and a_t is an idd $(0,\sigma_a^2)$ random variable. Specifically, we will consider ARMA(2,2) models since two degrees of freedom are required to allow the roots of the polynomial equations associated with $\theta(B)$ and $\phi(B)$ to be complex.

This model is written mathematically as

$$y_t = \frac{(1-\theta_1 B-\theta_2 B^2)}{(1-\phi_1 B-\phi_2 B^2)} a_t \qquad \text{(Eq. 3-2)}$$

which means

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} \qquad \text{(Eq. 3-3)}$$

Spectrum of a Stochastic Model

The spectrum, denoted by $f_y(\theta)$, of the output, y_t, of a linear filter, $\psi(B)$, is related to the spectrum, $f_a(\theta)$, of the input, a_t, as follows, Anderson (1971):

$$f_y(\theta) = |\psi(e^{-i\theta})|^2 \, f_a(\theta) \qquad \text{(Eq. 3-4)}$$

Since the input, a_t, is white noise, $f_a(\theta) = \frac{\sigma_a^2}{\pi}$. Thus,

$$f_y(\theta) = \frac{\sigma_a^2}{\pi} \left| \psi(e^{-i\theta}) \right|^2 \qquad \text{(Eq. 3-5)}$$

That is, the spectrum of $\{y_t\}$ is equal to the variance of a_t divided by π times the complex absolute value squared of the transfer function $\psi(B)$ evaluated at $B = e^{-i\theta}$.

Now,

$$\psi(B) = \frac{\theta(B)}{\phi(B)}$$

$$= \frac{(1-\theta_1 B-\theta_2 B^2)}{(1-\phi_1 B-\phi_2 B^2)} \qquad \text{(Eq. 3-6)}$$

can be written as

$$\psi(B) = \frac{(1-\nu B)(1-\bar{\nu} B)}{(1-\lambda B)(1-\bar{\lambda} B)} \qquad \text{(Eq. 3-7)}$$

where it is assumed that the roots of the associated polynomial equations of $\theta(B)$ and $\phi(B)$ are the complex conjugates ν, $\bar{\nu}$ and λ, $\bar{\lambda}$, respectively. The complex numbers ν and $\bar{\nu}$ will be called the zeros, and λ and $\bar{\lambda}$ will be called the poles of the ARMA(2,2) model. The zeros and poles are assumed to be complex conjugate pairs in order to allow them to be near $e^{i\omega}$, which will be shown in Chapter 4 to be necessary for spectral equivalence.

Therefore,

$$|\psi(e^{-i\theta})|^2 = \frac{|(1-\nu e^{-i\theta})(1-\bar{\nu} e^{-i\theta})|^2}{|(1-\lambda e^{-i\theta})(1-\bar{\lambda} e^{-i\theta})|^2} \qquad \text{(Eq. 3-8)}$$

If we multiply each factor in the numerator and denominator of Eq. 3-8 by $e^{i2\theta}$ we get

$$|\psi(e^{-i\theta})| = \frac{|(e^{i\theta} - \nu)(e^{i\theta} - \bar{\nu})|^2}{|(e^{i\theta} - \lambda)(e^{i\theta} - \bar{\lambda})|^2} \qquad \text{(Eq. 3-9)}$$

which can be rewritten, Churchill (1974), as

$$|\psi(e^{-i\theta})|^2 = \left[\frac{|e^{i\theta}-\nu|}{|e^{i\theta}-\lambda|}\right]^2 \cdot \left[\frac{|e^{i\theta} - \bar{\nu}|}{|e^{i\theta} - \bar{\lambda}|}\right]^2 \qquad \text{(Eq. 3-10)}$$

Figure 3-1 graphically illustrates Eq. 3-10 in the complex plane. Since $|e^{i\theta}-\nu|$ is the distance between $e^{i\theta}$ and ν, and $|e^{i\theta}-\lambda|$ is the distance between $e^{i\theta}$ and λ, the first term is equal to the ratio of these two lengths squared. Similarly, $|e^{i\theta}-\overline{\nu}|$ and $|e^{i\theta}-\overline{\lambda}|$ represent the distance between $e^{i\theta}$ and $\overline{\nu}$ and $\overline{\lambda}$, respectively. Hence the second term is equal to the ratio of these two lengths squared.

If we define ν and $\overline{\nu}$ such that in polar form

$$\nu = se^{i\sigma} \quad , \quad \overline{\nu} = se^{-i\sigma} \qquad \text{(Eq. 3-11)}$$

and λ and $\overline{\lambda}$ such that

$$\lambda = re^{i\rho} \quad , \quad \overline{\lambda} = re^{-i\rho} \qquad \text{(Eq. 3-12)}$$

then $|\psi(e^{-i\theta})|^2$, Eq. 3-8, can also be represented conveniently as

$$|\psi(e^{-i\theta})|^2 = \frac{|(1-se^{-i(\theta-\sigma)})(1-se^{-i(\theta+\sigma)})|^2}{|(1-re^{-i(\theta-\rho)})(1-re^{-i(\theta+\rho)})|^2}$$

$$= \frac{(1-2s\ \cos(\theta-\sigma)+s^2)}{(1-2r\ \cos(\theta-\rho)+r^2)} \cdot \frac{(1-2s\ \cos(\theta+\sigma)+s^2)}{(1-2r\ \cos(\theta+\rho)+r^2)} \qquad \text{(Eq. 3-13)}$$

The two factors in Eq. 3-10 and Eq. 3-13 can be seen to be equal by considering $|e^{i\theta}-\nu|$ as shown in Figure 3-2. Application of the Cosine formula for two sides and the included angle yields immediately:

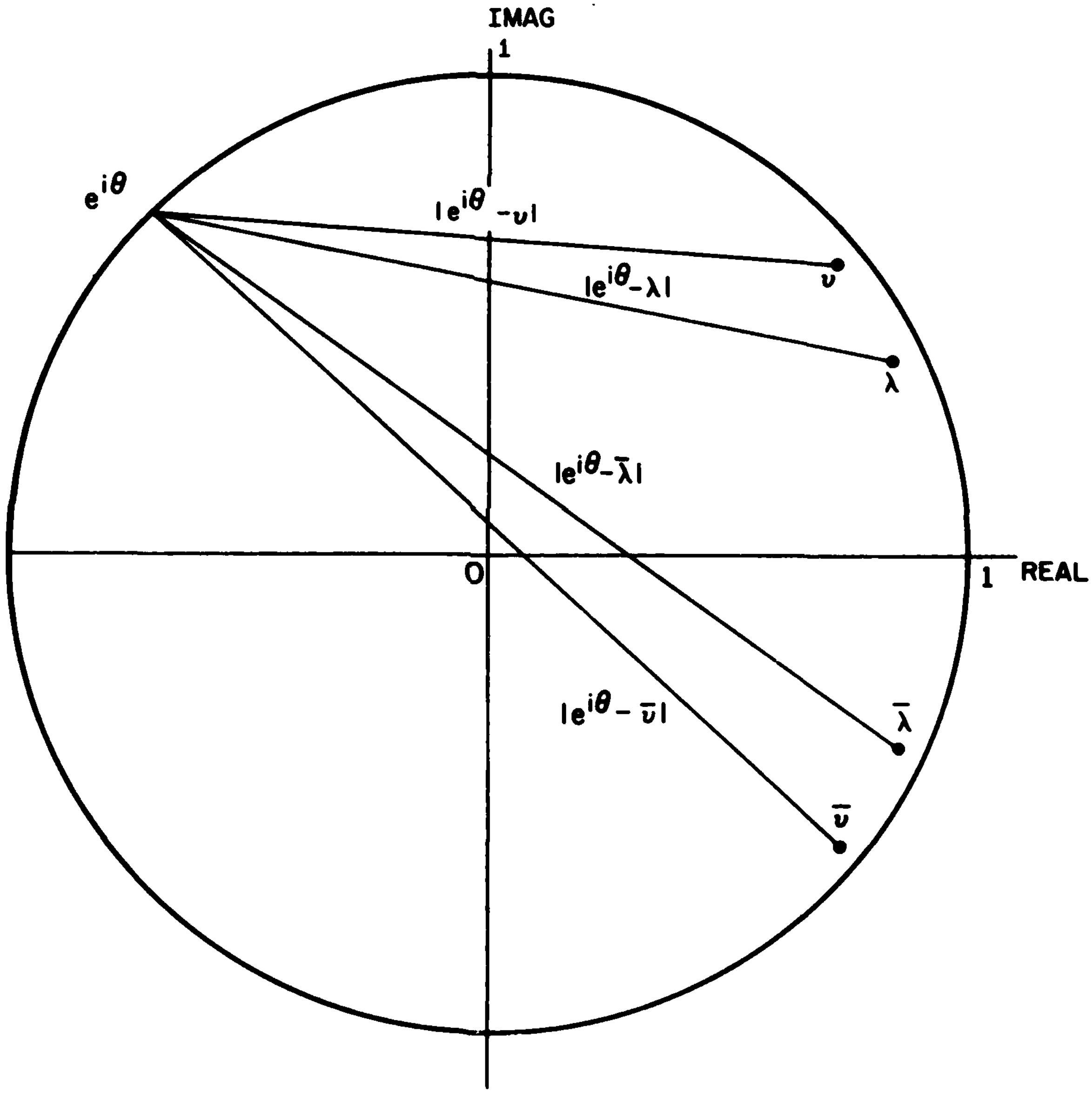

Figure 3-1. Graphical Interpretation of $|\psi(e^{-i\theta})|^2$

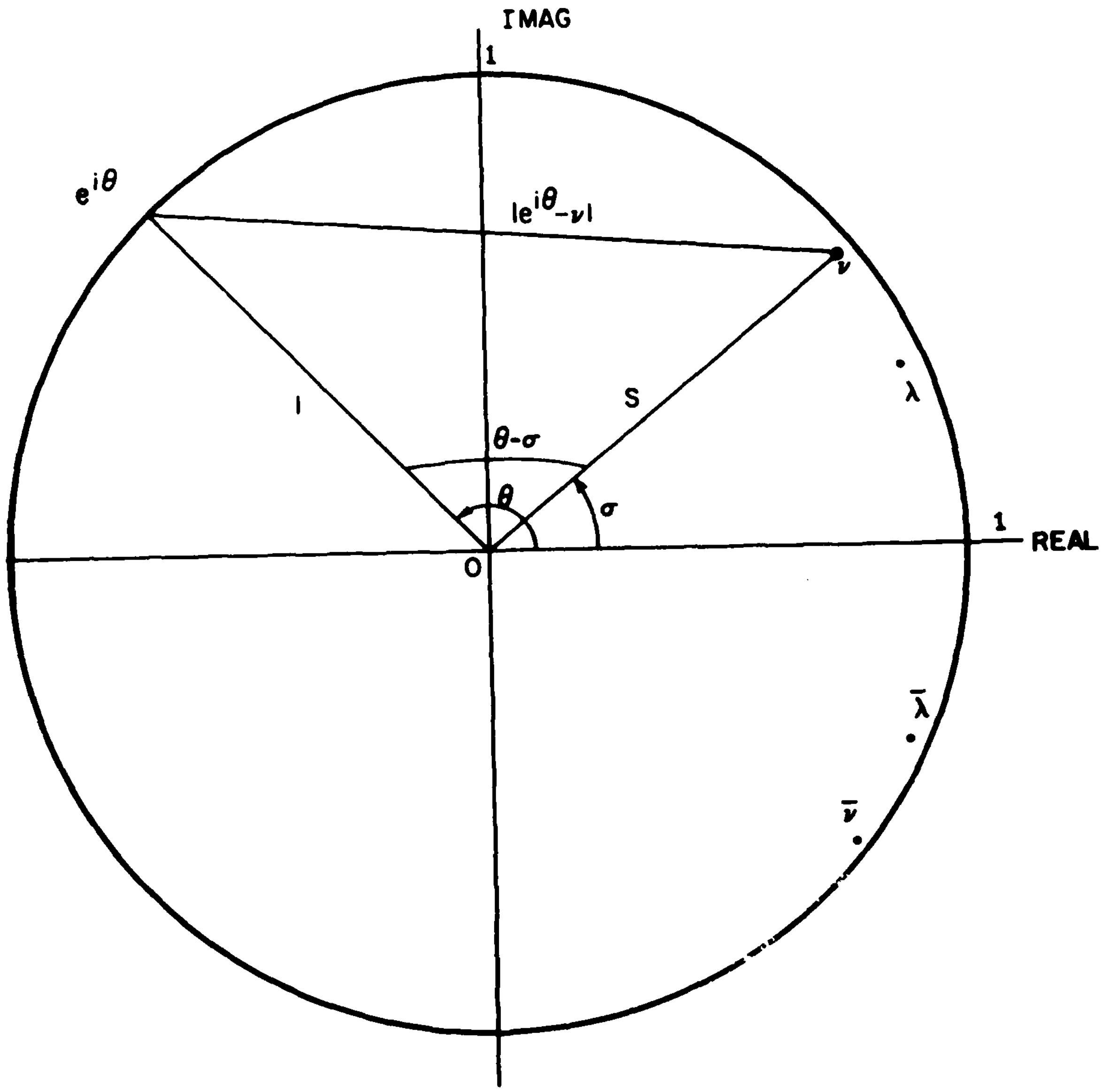

Figure 3-2. Trigonometric Development of $|e^{i\theta}-\nu|$

$$|e^{i\theta}-\nu|^2 = 1 + s^2 - 2s\cos(\theta-\sigma) \qquad \text{(Eq. 3-14)}$$

Similar application of the Cosine formula to the three other distances completes the demonstration.

In this chapter we have defined the spectrum and developed a graphical interpretation of the poles and zeros for a stochastic model. This interpretation will be used in Chapter 4 to identify equivalent deterministic and stochastic models.

4. EQUIVALENT DETERMINISTIC AND STOCHASTIC MODELS

Definition of Equivalence

Equivalence between the deterministic and stochastic models will be established via their spectral representations. The spectrum of the stochastic model will be equated to the spectrum of the deterministic model in the sense that they both represent the same frequency decomposition of variance. Since the area under the spectrum represents variance, equivalence is established by equating the area under each spectrum.

The area under the deterministic spectrum is found via the rectangular rule for integration. This is due to the nature of the discrete Fourier transform that was used to derive the deterministic spectrum. Brigham (1974) presents a nice graphical development of the discrete Fourier transform for sampled data. The rectangular area around any specific discrete frequency, say θ^*, represents the variance of x_t that is contained in the frequency interval $\theta^* - \frac{\pi}{N}$ to $\theta^* + \frac{\pi}{N}$. That is,

$$\mathrm{Var}(x_t)\Big|_{\theta^* \pm \frac{\pi}{N}} = \frac{2\pi}{N} f_x(\theta^*) \qquad \text{(Eq. 4-1)}$$

It is important to note, according to sampling theory, Kuo (1977), that a discrete time series can only contain information about the total variance in each of the $\frac{N}{2}$ frequency intervals of length $\frac{2\pi}{N}$. Thus the

spectrum of the discrete deterministic model represents the total variance in each interval but contains no information on how the variance is distributed within each frequency interval.

The spectrum for a stochastic model, however, is continuous. Therefore, the value of the spectrum, $f_y(\theta^*)$, at the frequency θ^* represents the variance of y_t at θ^*. The total variance in any interval is found by integrating $f_y(\theta)$ over the interval. Thus,

$$\mathrm{Var}(y_t)\Big|_{\theta^* \pm \frac{\pi}{N}} = \int_{\theta^* - \frac{\pi}{N}}^{\theta^* + \frac{\pi}{N}} f_y(\theta)d\theta \qquad \text{(Eq. 4-2)}$$

In order to directly compare the spectral representation of a stochastic model to the discrete spectrum of a deterministic model, we will define the equivalent discrete stochastic spectrum, denoted by $f_y^*(\theta)$, as follows:

$$f_y^*(\theta) = \int_{\theta - \frac{\pi}{N}}^{\theta + \frac{\pi}{N}} f_y(\theta)d\theta \qquad \text{(Eq. 4-3)}$$

where it is understood that $\theta = \frac{2\pi j}{N} \quad j=0,1,2,\ldots,\frac{N}{2}$.

In summary, the deterministic and stochastic models are equivalent when the frequency decomposition of the variance is the same for both models. That is, when $\frac{2\pi}{N} f_x(\theta) = f_y^*(\theta) \quad \theta = \frac{2\pi j}{N} \quad j = 0,1,2\ldots, \frac{N}{2}$.

Care must be taken for non-normal processes since the uniqueness implied above is only guaranteed for second order stationary normal processes, Jenkins and Watts (1968) and Chattfield (1975).

Procedure for Identifying Equivalent Models

Model equivalence requires interval by interval equality for each of the $\frac{N}{2}$ intervals; therefore, model equivalence also implies equality of the total variance over the entire frequency range from 0 to π. The equality of the total variance is a necessary, but not sufficient, condition for model equivalence. However, it can be used as a gross check on model equivalency.

Two concepts will now be introduced to provide a practical procedure for identifying an equivalent stochastic model for a given deterministic model.

The first concept relies on the geometric interpretation of the stochastic spectrum introduced in Chapter 3 and the normalized deterministic spectrum from Chapter 2.

A pair of complex conjugate poles (λ, $\bar{\lambda}$) and zeros (ν, $\bar{\nu}$) are shown in the unit circle on the complex plane in Figure 4-1. A graphical interpretation of $f_y(\theta)$ which was demonstrated in Chapter 3 (Eq. 3-5 and 3-10) to be a function of

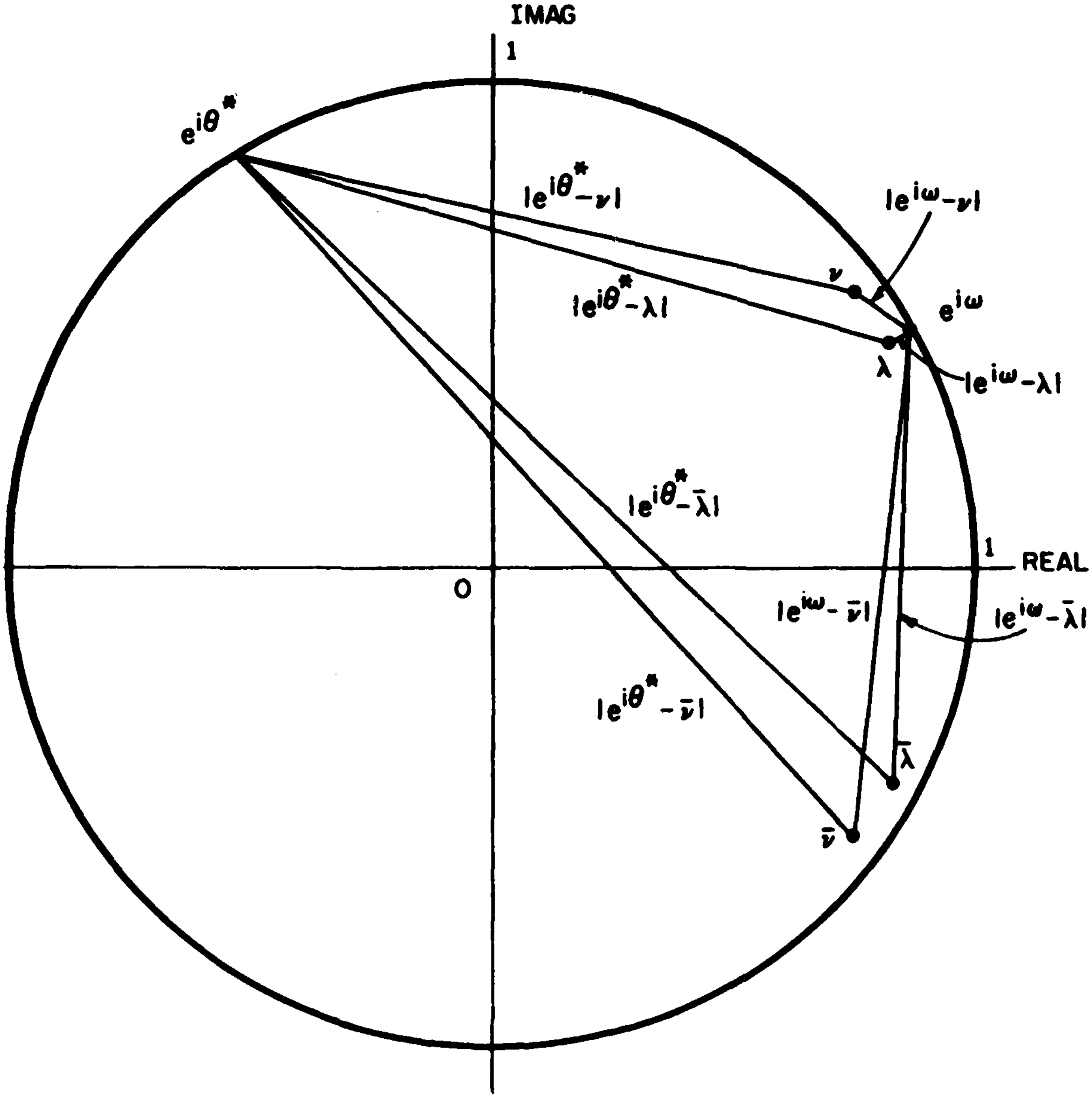

Figure 4-1. Graphical Interpretation of $|\psi(e^{-i\theta^*})|^2$ and $|\psi(e^{-i\omega})|^2$

$$|\psi(e^{-i\theta})|^2 = \left[\frac{|e^{i\theta} - \nu|}{|e^{i\theta} - \lambda|}\right]^2 \cdot \left[\frac{|e^{i\theta} - \bar{\nu}|}{|e^{i\theta} - \bar{\lambda}|}\right]^2 \qquad \text{(Eq. 4-4)}$$

is shown for $\theta = \theta^*$ and $\theta = \omega$.

It can be seen if λ and ν are close to the boundary of the unit circle, and near $e^{i\omega}$, with ν farther from $e^{i\omega}$ than λ is from $e^{i\omega}$, that when $\theta = \theta^*$

$$|e^{i\theta^*} - \nu| \doteq |e^{i\theta^*} - \lambda| \qquad \text{(Eq. 4-5)}$$

and

$$|e^{i\theta^*} - \bar{\nu}| \doteq |e^{i\theta^*} - \bar{\lambda}| \qquad \text{(Eq. 4-6)}$$

hence, from Eqs. 4-4, 4-5, and 4-6

$$|\psi(e^{-i\theta^*})|^2 \approx 1^2 \cdot 1^2 \doteq 1 \qquad \text{(Eq. 4-7)}$$

When $\theta = \omega$

$$|e^{i\omega} - \nu| >> |e^{i\omega} - \lambda| \qquad \text{(Eq. 4-8)}$$

and

$$|e^{i\omega} - \bar{\nu}| \doteq |e^{i\omega} - \bar{\lambda}| \qquad \text{(Eq. 4-9)}$$

hence, from Eqs. 4-4, 4-8, and 4-9

$$|\psi(e^{-i\omega})|^2 \doteq \left[\frac{|e^{i\omega} - \nu|}{|e^{i\omega} - \lambda|}\right]^2 \cdot 1^2 \approx \left[\frac{|e^{i\omega} - \nu|}{|e^{i\omega} - \lambda|}\right]^2 \qquad \text{(Eq. 4-10)}$$

Thus, under these assumptions, the stochastic spectrum

$$f_y(\theta) = \frac{\sigma_a^2}{\pi} \left|\psi(e^{-i\theta})\right|^2 \qquad \text{(Eq. 4-11)}$$

is approximately equal to $\frac{\sigma_a^2}{\pi}$ for $\theta \neq \omega$ and approximately equal to

$\frac{\sigma_a^2}{\pi} \frac{|e^{i\theta}-\nu|^2}{|e^{i\theta}-\lambda|^2}$ for $\theta = \omega$.

The most critical case in the above approximation occurs when θ^* is near ω. We want $|e^{i\omega}-\nu|$ much larger than $|e^{i\omega}-\lambda|$ and, simultaneously, $|e^{i\theta^*}-\nu|$ approximately equal to $|e^{i\theta^*}-\lambda|$. This means that $|e^{i\omega}-\nu|$ must be much smaller than $|e^{i\theta^*}-e^{i\omega}|$ for θ^* near ω. Since

$$f_y^*\left(\omega - \frac{2\pi}{N}\right) = \int_{\omega-\frac{3\pi}{N}}^{\omega-\frac{\pi}{N}} f_y(\theta)d\theta \qquad \text{(Eq. 4-12)}$$

and

$$f_y^*\left(\omega + \frac{2\pi}{N}\right) = \int_{\omega+\frac{\pi}{N}}^{\omega+\frac{3\pi}{N}} f_y(\theta)d\theta \qquad \text{(Eq. 4-13)}$$

the most critical value for θ^* is when $\theta^* = \omega \pm \frac{\pi}{N}$. Hence, we want

$$|e^{i\omega} - \nu| << |e^{i(\omega\pm\pi/N)} - e^{i\omega}| = 2 \sin \frac{\pi}{2N} \qquad \text{(Eq. 4-14)}$$

The spectrum of a deterministic sinusoid plus white noise, i.e.,

$$x_t = A \cos(\omega t + \omega_o) + a_t \qquad \text{(Eq. 4-15)}$$

where

a_t is $\mathrm{iid}(0,\sigma_a^2)$

a_t and d_t are independent

was shown in Chapter 2, Eq. 2-17, to be

$$f_x(\theta) = \begin{cases} \dfrac{A^2}{2} \cdot \dfrac{N}{2\pi} + \dfrac{\sigma_a^2}{\pi} & \theta = \omega \\ \\ \dfrac{\sigma_a^2}{\pi} & \theta \neq \omega \end{cases} \qquad \text{(Eq. 4-16)}$$

Figure 4-1 has shown that for suitably chosen λ and ν that

$$f_y(\theta) \approx \begin{cases} \dfrac{\sigma_a^2}{\pi} \left[\dfrac{|e^{i\omega} - \nu|}{|e^{i\omega} - \lambda|}\right]^2 & \theta = \omega \\ \\ \dfrac{\sigma_a^2}{\pi} & \theta \neq \omega \end{cases} \qquad \text{(Eq. 4-17)}$$

Thus the two spectra represent the same general decomposition of variance.

The second concept for identifying the specific form of the stochastic spectrum takes advantage of the general form of $f_y(\theta)$ and the equality of total variance over the entire spectra.

If the total area under each spectrum is equal and the spectra are equal everywhere except for $\theta = \omega$, then the area in the interval around ω must be equal. That is, if

$$\left(\frac{A^2}{2} \cdot \frac{N}{2\pi} + \frac{\sigma_a^2}{\pi}\right) \cdot \left(\frac{2\pi}{N}\right) + \frac{\sigma_a^2}{\pi}\left(\frac{2\pi}{N}\right)\left(\frac{N}{2} - 1\right) = \int_0^{\pi} f_y(\theta)\, d\theta \quad \text{(Eq. 4-18)}$$

$$\frac{A^2}{2} + \sigma_a^2 = \int_0^{\pi} f_y(\theta)\, d\theta \quad \text{(Eq. 4-19)}$$

and

$$f_x(\theta) = f_y(\theta) \qquad \theta \neq \omega \quad \text{(Eq. 4-20)}$$

then

$$\frac{A^2}{2} + \frac{2\sigma_a^2}{N} = \int_{\theta = \omega - \frac{\pi}{N}}^{\theta = \omega + \frac{\pi}{N}} f_y(\theta)\, d\theta \quad \text{(Eq. 4-21)}$$

The identification procedure, therefore, consists of, 1) choosing λ and ν such that $f_y(\theta)$ has the general form of Eq. 4-17 with $\int_0^\pi f_y(\theta)d\theta = \frac{A^2}{2} + \sigma_a^2$ and, 2) comparing $f_y^*(\theta)$ to $\frac{2\pi}{N} f_x(\theta)$ for $\theta = \frac{2\pi j}{N}$ $j = 0,1,2,\ldots, \frac{N}{2}$.

Before discussing specific procedures for finding λ and ν a method for evaluating the $\int_0^\pi f_y(\theta)d\theta$ utilizing Green's function will be presented. This will eliminate the need to do some integrations in the identification procedure.

Green's function, also called ψ-weights by Box-Jenkins (1970), represents an orthogonal decomposition of a time series by expressing the time series as a linear combination of independent random variables. That is, for

$$y_t = \frac{\theta(B)}{\phi(B)} a_t = \psi(B)a_t \qquad \text{(Eq. 4-22)}$$

Green's function, denoted by G_j, is defined such that

$$y_t = \sum_{j=0}^{\infty} G_j a_{t-j} \qquad \text{(Eq. 4-23)}$$

Since the a_t's are mutually independent, identically distributed random variables, Eq. 4-23 implies, Hogg & Craig (1978):

$$\sigma_y^2 = \sigma_a^2 \sum_{j=0}^{\infty} G_j^2 \qquad \text{(Eq. 4-24)}$$

where σ_y^2 is the total variance of the output of the linear process, $\psi(B)$, when driven by white noise of variance σ_a^2. In other words,

$$\sigma_y^2 = \int_0^{\pi} f_y(\theta)\, d\theta \qquad \text{(Eq. 4-25)}$$

It can be shown, Appendix C, for an ARMA(2,2) model with complex poles and zeros, that

$$\sum_{j=0}^{\infty} G_j^2 = 1 + \frac{R^2 r^2}{2}\left[\frac{\cos(2(\rho-\beta)) - r^2 \cos 2\beta}{1 + r^4 - 2r^2 \cos 2\rho} + \frac{1}{1-r^2}\right] \qquad \text{(Eq. 4-26)}$$

where $r = |\lambda|$

$\rho = \mathrm{Arg}\ (\lambda)$

$$R = \sqrt{\left(1-\frac{\theta_2}{\phi_2}\right)^2 + B^2}$$

$$B = \frac{\phi_1 - 2\theta_1 + \dfrac{\phi_1\theta_2}{\phi_2}}{\sqrt{-(\phi_1^2 + 4\phi_2)}}$$

$$\beta = \tan^{-1}\left[\frac{B}{1 - \frac{\theta_2}{\phi_2}}\right]$$

The spectral equivalence of deterministic and stochastic models has been developed in this chapter. A procedure to identify an equivalent stochastic model for a given deterministic model also has been presented. This procedure utilizes the graphical interpretation of the stochastic spectrum presented in Chapter 3 and the analytic expression for $\sum_{j=0} G_j^2$ derived in Appendix C. The identification procedure will be illustrated in the next chapter.

5. EXAMPLE OF THE IDENTIFICATION OF EQUIVALENT MODELS

General Discussion

The following example illustrates the specifics of the procedure for identifying an equivalent stochastic model for a given deterministic model. The deterministic model is "given" in the sense that Fourier analysis of $\{x_t\}$ has been used to identify the deterministic model. The converse problem -- given a stochastic model: find the equivalent deterministic model -- is solved by a straightforward (actually simpler) application of the principles illustrated in this example.

Given

$$x_t = d_t + a_t \qquad t = 1, 2, \ldots, N \qquad \text{(Eq. 5-1)}$$

where

$$d_t = A\cos(\omega t + \omega_o)$$

a_t is an iid $(0,\sigma_a^2)$ random variable

and

d_t and a_t are independent

Find

$$y_t = \frac{(1-\theta_1 B-\theta_2 B^2)}{(1-\phi_1 B-\phi_2 B^2)}\ a_t \qquad \text{(Eq. 5-2)}$$

such that $\{x_t\}$, $\{y_t\}$ have the same second order properties

While the choice of λ and ν is restricted to be near $e^{i\omega}$ and inside the unit circle, this still leaves an infinity from which to choose. However, the "best" placement for both λ and ν is such that their arguments are both equal to ω.

Define λ and ν such that in polar form

$$\lambda = r\, e^{i\rho} \qquad \text{(Eq. 5-3)}$$

$$\nu = s\, e^{i\sigma} \qquad \text{(Eq. 5-4)}$$

and let

$$\rho = \sigma = \omega \qquad \text{(Eq. 5-5)}$$

This placement is "best" in the sense that the spectrum evaluated at the discrete frequency before and after ω will be approximately equal. That is,

$$f_y^*(\omega - \frac{2\pi}{N}) \approx f_y^*(\omega + \frac{2\pi}{N}) \qquad \text{(Eq. 5-6)}$$

This means that both of these values can simultaneously be made arbitrarily close to $\frac{2\sigma_a^2}{N}$. Reca.1, Eq. 4-17, that

$$f_y(\omega) \approx \frac{\sigma_a^2}{\pi} \left[\frac{|e^{i\omega}-\nu|}{|e^{i\omega}-\lambda|}\right]^2 \qquad \text{(Eq. 5-7)}$$

Thus, $f_y(\omega)$ is dominated by the ratio of the distance between $e^{i\omega}$ and ν and the distance between $e^{i\omega}$ and λ. This is also true for the spectrum near ω. Hence,

$$f_y^*(\omega - \frac{2\pi}{N}) \approx f_y^*(\omega + \frac{2\pi}{N}) \approx \frac{2\sigma_a^2}{N} \qquad \text{(Eq. 5-8)}$$

requires that

$$\frac{|e^{i(\omega \pm \pi/N)} - \nu|}{|e^{i(\omega \pm \pi/N)} - \lambda|} \approx 1 \qquad \text{(Eq. 5-9)}$$

This condition will be satisfied when

$$|e^{i\omega} - \nu| \text{ and } |e^{i\omega} - \lambda| < < |e^{i(\omega \pm \pi/N)} - e^{i\omega}| = 2 \sin \frac{\pi}{2N} \qquad \text{(Eq. 5-10)}$$

The above condition is satisfied when both λ and ν are much closer to $e^{i\omega}$ than to $e^{i(\omega \pm \pi/N)}$. When $\rho = \sigma = \omega$, both λ and ν are equally far from $e^{i(\omega - \pi/N)}$ and $e^{i(\omega + \pi/N)}$, respectively, and the ratio of the distances can be made arbitrarily close to one. This idea is illustrated in Figure 5-1. The distance between $e^{i\omega}$ and $e^{i(\omega \pm \pi/N)}$ has been greatly exaggerated to illustrate the concept of relative "nearness"/"farness." As can be seen in Figure 5-1,

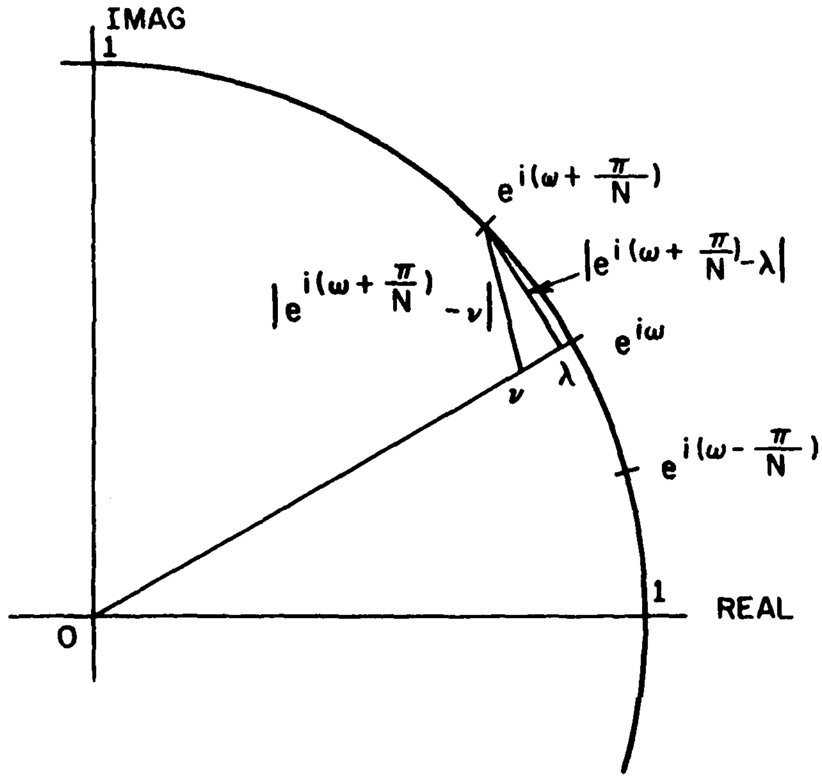

Figure 5-1. Graphical Interpretation of $|\psi(e^{-i(\omega \pm \pi/N)})|^2$

$$\frac{|e^{i(\omega \pm \pi/N)} - \nu|}{|e^{i(\omega \pm \pi/N)} - \lambda|} \approx 1 \qquad \text{(Eq. 5-11)}$$

even though

$$\frac{|e^{i\omega} - \nu|}{|e^{i\omega} - \lambda|} = \frac{1 - s}{1 - r} >> 1 \qquad \text{(Eq. 5-12)}$$

Thus, placement of λ and ν on the same radian as ω with $r > s$ results in a stochastic spectrum of the same general form as the deterministic spectrum.

To find specific values for λ and ν, given $\rho = \sigma = \omega$, we will assign r arbitrarily *close to one* and calculate the corresponding value of s such that

$$\int_0^{\pi} f_y(\theta)\, d\theta = \frac{A^2}{2} + \sigma_a^2 \qquad \text{(Eq. 5-13)}$$

via

$$\sum_{j=0}^{\infty} G_j^2 \qquad \text{(Eq. 5-14)}$$

Numerical Example

The identification procedure will now be illustrated with a numerical example. Assume that Fourier analysis has been used to identify the following sinusoidal plus white noise model; this example is adapted from O'Connor and Kapoor (1982).

Given:

$$A = 8.643$$

$$\omega = \frac{2\pi}{24}$$

$$\sigma_a^2 = 8.460 \qquad \text{(Eq. 5-15)}$$

$$N = 240$$

The deterministic spectrum is

$$f_x(\theta) = \begin{cases} \dfrac{A^2}{2} \cdot \dfrac{N}{2\pi} + \dfrac{\sigma_a^2}{\pi} = 1{,}429.4 & \theta = \dfrac{2\pi}{24} \\[2ex] \dfrac{\sigma_a^2}{\pi} = 2.693 & \theta \neq \dfrac{2\pi}{24} \end{cases} \qquad \text{(Eq. 5-16)}$$

$$\text{for } \theta = \frac{2\pi j}{240} \qquad j = 0, 1, \ldots, 120$$

$$\text{and} \quad \sigma_x^2 = \frac{A^2}{2} + \sigma_a^2 = 45.811$$

Therefore the frequency decomposition of the variance of x_t, obtained by integrating $f_x(\theta)$ via the rectangular rule, is

$$\mathrm{Var}(x_t)\Big|_{\theta\pm\frac{\pi}{N}} = \frac{2\pi}{N} f_x(\theta) = \begin{cases} \frac{A^2}{2} + \frac{2\sigma_a^2}{N} = 37.421 & \theta = \omega \\ \frac{2\sigma_a^2}{N} \qquad = \quad .0705 & \theta \neq \omega \end{cases} \qquad \text{(Eq. 5-17)}$$

Let $\lambda = .999\ e^{i\frac{2\pi}{24}}$ **(Eq. 5-18)**

and calculate $\nu = s\ e^{i\frac{2\pi}{24}}$ such that

$$\sigma_a^2 \sum_{j=0}^{\infty} G_j^2 = \frac{A^2}{2} + \sigma_a^2 = 45.811 \qquad \text{(Eq. 5-19)}$$

i.e.

$$\sum_{j=0}^{\infty} G_j = \frac{A^2}{2} \cdot \frac{1}{\sigma_a^2} + 1 = 5.415 \qquad \text{(Eq. 5-20)}$$

An iterative solution of Eq. 4-26 gives s = .9308 when $\sum_{j=0}^{\infty} G_j^2 = 5.415$.

Thus

$$\lambda = 0.999\ e^{i\frac{2\pi}{24}} \qquad \nu = 0.9308\ e^{i\frac{2\pi}{24}} \qquad \text{(Eq. 5-21)}$$

and therefore

$$\phi_1 = \lambda + \bar{\lambda} = 1.9299 \qquad \theta_1 = \nu + \bar{\nu} = 1.7982$$

$$\phi_2 = -\lambda\bar{\lambda} = -0.9980 \qquad \theta_2 = -\nu\bar{\nu} = -0.8664 \qquad \text{(Eq. 5-22)}$$

$$\sigma_y^2 = \sigma_a^2 \sum_{j=0}^{\infty} G_j^2 = 45.811$$

Comparing

$$|e^{i\omega} - \nu| = 1 - s = .069 \qquad \text{(Eq. 5-23)}$$

to

$$|e^{i(\omega \pm \pi/N)} - e^{i\omega}| = 2 \sin \frac{\pi}{2N} = .013 \qquad \text{(Eq. 5-24)}$$

shows that ν is not much closer to $e^{i\omega}$ than the distance between $e^{i\omega}$ and $e^{i(\omega \pm \pi/N)}$. The effect of this can be seen in Table 1, which shows a portion of $f_y(\theta)$ and the $\int_{\theta - \frac{\pi}{N}}^{\theta + \frac{\pi}{N}} f_y(\theta)d\theta$. The variance of y_t around ω, $f_y^*(\omega)$, is 36.661 vs $\frac{2\pi}{N} f_x(\omega) = 37.421$. Also the variance of y_t in the intervals on either side of ω is large compared to the desired $\frac{2\pi}{N} f_x(\omega \pm 2\pi/N) =$ .0705 while the variance at high frequencies is low. The deterministic model has a total variance, $\sigma_x^2 = 45.811$, distributed such that the variance of x_t at ω is 37.421 and the remaining variance of 8.390 is distributed evenly at the remaining frequencies. This stochastic model results in a total variance, $\sigma_y^2 = 45.811$, but the variance of y_t at ω is 36.661, which is 80.0% of the total variance versus the desired 81.7%. Also the variance of y_t at $\omega \pm \frac{2\pi}{N}$ is about .68, or almost 10 times the desired variance. Therefore, this stochastic model is not exactly equivalent to the given deterministic model.

We can improve the approximation by moving both λ and ν closer to the unit circle. When $\lambda = .99999\, e^{i\frac{2\pi}{24}}$, we find $\nu = .9933\, e^{i\frac{2\pi}{24}}$. Hence

Table 5-1. Stochastic Spectrum: r=.999, s=.9308

j	$\theta = \frac{2\pi j}{N}$	$f_y(\theta)$	$\int f_y(\theta)d\theta$
0	0.000	2.704	.0354
1	.026	2.715	.0711
2	.052	2.750	.0720
3	.079	2.818	.0738
4	.105	2.935	.0769
5	.131	3.143	.0825
6	.157	3.537	.0931
7	.183	4.400	.1167
8	.209	6.880	.1879
9	.236	20.271	.6857
10	.262	12240.423	36.6606
11	.288	20.197	.6836
12	.314	6.829	.1866
13	.340	4.350	.1154
14	.367	3.481	.0916
15	.393	3.078	.0808
16	.419	2.859	.0749
17	.445	2.726	.0714
18	.471	2.639	.0691
19	.497	2.579	.0675
20	.524	2.536	.0664
21	.550	2.504	.0656
22	.576	2.480	.0649
23	.602	2.461	.0644
24	.628	2.445	.0640
25	.654	2.433	.0637
26	.681	2.422	.0634
27	.707	2.414	.0632
28	.733	2.406	.0630
29	.759	2.400	.0628
30	.785	2.395	.0627
⋮	⋮	⋮	⋮
119	3.115	2.344	.0614
120	3.142	2.344	.0307

$$\phi_1 = 1.93183$$
$$\phi_2 = -0.99998$$
$$\theta_1 = 1.91895 \qquad \text{(Eq. 5-25)}$$
$$\theta_2 = -0.98669$$
$$\sigma_y^2 = 45.811$$

In this case ν is close to $e^{i\omega}$, .007, compared to the distance between $e^{i\omega}$ and $e^{i(\omega \pm \pi/N)}$, .013. Table 2 also shows that the distribution of the variance of y_t is very close to the desired variance distribution of x_t. The variance of y_t at ω, 37.515, is 81.9% of the total variance versus the desired variance of 37.421 or 81.7%, and the remainder of the variance is quite evenly distributed.

A stochastic model which represents the same frequency decomposition of variance, to any arbitrary approximation, can be found for any real valued sinusoid plus white noise. The required degree of approximation depends on the application.

The identification of an equivalent, or approximately equivalent, deterministic model for a given stochastic model relies on the same principles illustrated in the above example. The "goodness" of the approximation is evaluated by considering the frequency decomposition of variance.

For example, suppose we have identified as the best stochastic model for a given time series the ARMA(2,2) whose spectrum is shown in Table 1. The concentration of the variance of y_t in the interval

Table 5-2. Stochastic Spectrum: r=.99999, s=.9933

j	$\theta = \frac{2\pi j}{N}$	$f_y(\theta)$	$\int f_y(\theta)d\theta$
0	0.000	2.661	.0348
1	.026	2.661	.0697
2	.052	2.661	.0697
3	.079	2.662	.0697
4	.105	2.663	.0697
5	.131	2.665	.0698
6	.157	2.669	.0699
7	.183	2.677	.0701
8	.209	2.701	.0708
9	.236	2.832	.0757
10	.262	1192486.736	37.5145
11	.288	2.832	.0756
12	.314	2.701	.0708
13	.340	2.677	.0701
14	.367	2.668	.0699
15	.393	2.664	.0698
16	.419	2.662	.0697
17	.445	2.661	.0697
18	.471	2.660	.0696
19	.497	2.659	.0696
20	.524	2.659	.0696
21	.550	2.659	.0696
22	.576	2.659	.0696
23	.602	2.658	.0696
24	.628	2.658	.0696
25	.654	2.658	.0696
26	.681	2.658	.0696
27	.707	2.658	.0696
28	.733	2.658	.0696
29	.759	2.658	.0696
30	.785	2.658	.0696
⋮	⋮	⋮	⋮
119	3.115	2.657	.0696
120	3.142	2.657	.0348

$(\frac{2\pi}{24} - \frac{\pi}{240}, \frac{2\pi}{24} + \frac{\pi}{240})$ suggests the presence of a sinusoid whose period is 24. A hypothesis test of the "goodness" of modeling this time series as a sinusoid plus white noise can be based on the difference in the sum of the squared errors explained by the stochastic and deterministic models and the degree to which the residual series of the deterministic model departs from the white noise assumption.

This type of analysis provides the basis for determining the significance of pole-zero pairs. The significance of cancelling nearly equal poles and zeros or modeling their effect with a sinusoid can be determined through an analysis of variance.

This chapter has illustrated numerically the procedure for identifying equivalent deterministic and stochastic models for time series with a periodic component. We have shown that the significance of departures from exact equivalence for approximately equivalent deterministic and stochastic models can be evaluated by comparing the frequency decomposition of variance implied by each model.

6. SUMMARY AND CONCLUSIONS

Summary

The concept of equivalency between a second order stationary stochastic and deterministic representation of a periodic component in a time series was developed in Chapter 4. A deterministic and stochastic model are equivalent when they represent the same frequency decomposition of variance. Equivalence of the spectral representations requires integration of the continuous form of the stochastic spectrum over each of the $\frac{N}{2}$ intervals of length $\frac{2\pi}{N}$ to make the stochastic spectrum directly comparable to the discrete spectrum of the deterministic representation.

A procedure for finding an equivalent stochastic representation for a given deterministic representation that uses the graphical interpretation of the stochastic spectrum and the analytic expression for $\sum_{j=0}^{\infty} G_j^2$ to calculate the integral of the stochastic spectrum was also developed in Chapter 4 and illustrated in Chapter 5. This procedure consists of: 1) choosing the poles and zeros of the stochastic model such that the spectrum has a peak at the same frequency as the deterministic sinusoid and such that the total variance of the output of the stochastic process is equal to the total variance of the sinusoid plus white noise deterministic model; and 2) comparing the frequency decomposition of the variance of the stochastic model to that of the deterministic model for each discrete frequency interval.

Important implications of these results are as follows. The identification of mathematical equivalence, when it exists, allows the time series analyst to choose between potentially equivalent deterministic and stochastic representations. Both representations reflect the same information contained in the available realization of $\{x_t\}$ and are mathematically equivalent second order stationary models. Hence, there are no grounds, based on $\{x_t\}$, to argue for one representation versus its equivalent alternative. The choice must be based on considerations of the process itself (e.g., physical arguments) or on other information beyond that contained in $\{x_t\}$. This result provides a basis for identifying the similarities and differences among alternative time series models.

The concept of equivalence can also be used to evaluate models whose spectral representations, while not exactly equivalent, are approximately equivalent. Although any real valued sinusoid can be represented by an equivalent stochastic model, not all stochastic processes can be represented exactly by a single real valued sinusoid. The significance of the departure from exact equivalence can be evaluated by comparing the frequency decomposition of variance implied by each model.

These two concepts provide a way to identify deterministic sinusoids by using stochastic modeling techniques. This may prove particularly useful for time series which unexpectedly contain sinusoidal behavior or which contain sinusoids of unknown frequency. Conversely,

deterministic structure can be identified using Fourier spectral techniques and the results transformed to their stochastic equivalent.

This analysis also provides some insight into the role of pole-zero pairs in stochastic models. For example, it has been shown that pole-zero pairs near the unit circle are not necessarily indicative of overfitting, i.e., they do not necessarily cancel. These results provide criteria for determining when pole-zero pairs can be cancelled based on the variance associated with the pole-zero pair. If the associated variance is large, then the poles and zeros cannot be cancelled. If the variance is small, then the poles and zeros may be cancelled depending on the purpose of the model -- characterization, forecasting, or control -- and the understanding of the underlying process dynamics.

This insight looks promising for explaining several empirical observations, such as: the general requirement for a seasonal moving average term when seasonal differencing is used; the ability of AR(2) models to represent a sinusoidal determinism; and the ability of ARMA(2,1), ARMA(4,3),...,ARMA(2n,2n-1) models with pole-zero pairs near the unit circle to represent a sum of sinusoids plus a stochastic-only random element.

Conclusions

The following conclusions are drawn from this research:

1. Under the assumption of integer multiples of periodicity, the spectrum of a deterministic time series was shown, Chapter 2, to be particularly simple.

2. The graphical interpretation, presented in Chapter 3, of the poles and zeros in the unit circle of the complex plane provides useful insight to the stochastic spectrum.

3. The analytic expression for $\sum_{j=0}^{\infty} G_j^2$, derived in Appendix C, is helpful for identifying equivalent deterministic and stochastic models.

4. Spectrally equivalent deterministic and stochastic models can be identified through the identification procedure developed in Chapter 4 and illustrated in Chapter 5.

5. This research has identified the need for future work as follows:

i. The equivalency of the spectral representations in the frequency domain should be transformed to the time domain and interpreted in terms of the autocovariance function.

ii. The procedures for identifying an equivalent stochastic representation for a given deterministic representation should be extended to address a sum of sinusoids plus a stochastic only random element, i.e.,

$$\sum_{j=1}^{M} A_j \cos(\omega_j t + \omega_{oj}) + \psi(B)a_t = \psi'(B)a_t \qquad \text{(Eq. 6-1)}$$

iii. The procedures advocated by the three most commonly used time series modeling approaches -- Box-Jenkins, DDS, and deterministic plus stochastic -- should be examined and interpreted in the light of the above extensions.

APPENDIX A. SPECTRUM OF A SINUSOID

If

$$d_t = A \cos(\omega t + \omega_o) \qquad t = 1,2,\ldots, N \qquad \text{(Eq. A-1)}$$

then, by definition, the spectrum, $f_d(\theta)$ is

$$f_d(\theta) = \frac{1}{\pi N} \left| \sum_{t=1}^{N} d_t e^{i\theta t} \right|^2 \qquad 0 \leq \theta \leq \pi$$

$$= \frac{1}{\pi N} \left[\left(\sum_{t=1}^{N} d_t \cos \theta t \right)^2 + \left(\sum_{t=1}^{N} d_t \sin \theta t \right)^2 \right]$$

$$= \frac{A^2}{\pi N} \left[\left(\sum_{t=1}^{N} \cos(\omega t + \omega_o) \cos \theta t \right)^2 \right.$$

$$\left. + \left(\sum_{t=1}^{N} \cos(\omega t + \omega_o) \sin \theta t \right)^2 \right] \qquad \text{(Eq. A-2)}$$

The terms in each summation can be rewritten as

$$\cos(\omega t + \omega_o) \cos \theta t = \frac{1}{2} \left[\cos((\omega + \theta)t + \omega_o) \right.$$

$$\left. + \cos((\omega - \theta)t + \omega_o) \right] \qquad \text{(Eq. A-3)}$$

and

$$\cos(\omega t + \omega_o) \sin \theta t = \frac{1}{2} \left[\sin((\omega + \theta)t + \omega_o) \right.$$

$$\left. - \sin((\omega - \theta)t + \omega_o) \right] \qquad \text{(Eq. A-4)}$$

Also

$$\sum_{t=1}^{N} \cos\ (\alpha t+\beta) = \cos\left(\frac{(N+1)\alpha}{2}+\beta\right) \frac{\sin \frac{N\alpha}{2}}{\sin \frac{\alpha}{2}} \qquad \text{(Eq. A-5)}$$

which can be shown as follows

$$\sum_{t=1}^{N} \cos\ (\alpha t + \beta) = \frac{1}{2} \sum_{t=1}^{N} \left[e^{i(\alpha t + \beta)} + e^{-i(\alpha t + \beta)}\right]$$

$$= \frac{1}{2} e^{i\beta} \sum_{t=1}^{N} e^{i\alpha t} + \frac{1}{2} e^{-i\beta} \sum_{t=1}^{N} e^{-i\alpha t}$$

$$= \frac{1}{2} \left[e^{i\beta} \frac{e^{i\alpha}(1 - e^{iN\alpha})}{1 - e^{i\alpha}} + e^{-i\beta} \frac{e^{-i\alpha}\ (1 - e^{-iN\alpha})}{1 - e^{-i\alpha}}\right]$$

$$= \frac{1}{2} \left[e^{i(\alpha+\beta)} \frac{e^{iN\alpha/2}\left(e^{-iN\alpha/2}-e^{iN\alpha/2}\right)}{e^{i\alpha/2}\left(e^{-i\alpha/2}-e^{i\alpha/2}\right)}\right.$$

$$\left. + e^{-i(\alpha+\beta)} \frac{e^{-iN\alpha/2}\left(e^{iN\alpha/2}-e^{-iN\alpha/2}\right)}{e^{-i\alpha/2}\left(e^{i\alpha/2}-e^{-i\alpha/2}\right)}\right] \qquad \text{(Eq. A-6)}$$

Dividing both numerator and denominator by 2i and collecting terms yields

$$= \frac{1}{2} \left[e^{i\left(\frac{(N+1)\alpha}{2}+\beta\right)} \frac{\sin \frac{N\alpha}{2}}{\sin \frac{\alpha}{2}} + e^{-i\left(\frac{(N+1)\alpha}{2}+\beta\right)} \frac{\sin \frac{N\alpha}{2}}{\sin \frac{\alpha}{2}}\right] \qquad \text{(Eq. A-7)}$$

Finally

$$\sum_{t=1}^{N} \cos(\alpha t + \beta) = \cos\left(\frac{(N+1)\alpha}{2} + \beta\right) \frac{\sin \frac{N\alpha}{2}}{\sin \frac{\alpha}{2}} \qquad \text{(Eq. A-8)}$$

A similar derivation gives

$$\sum_{t=1}^{N} \sin(\alpha t + \beta) = \sin\left(\frac{(N+1)\alpha}{2} + \beta\right) \frac{\sin \frac{N\alpha}{2}}{\sin \frac{\alpha}{2}} \qquad \text{(Eq. A-9)}$$

Now the first term in Eq. A-2 though the use of Eq. A-3 and A-5 can be rewritten as

$$\sum_{t=1}^{N} A \cos(\omega t+\omega_o) \cos \theta t = \frac{A}{2} \sum_{t=1}^{N} \cos\left[(\omega+\theta)t+\omega_o\right]$$

$$+ \frac{A}{2} \sum_{t=1}^{N} \cos\left[(\omega-\theta)t+\omega_o\right]$$

$$= \frac{A}{2} \cos\left[\frac{(N+1)(\omega+\theta)}{2} + \omega_o\right] \frac{\sin \frac{N(\omega+\theta)}{2}}{\sin \frac{\omega+\theta}{2}}$$

$$+ \frac{A}{2} \cos\left[\frac{(N+1)(\omega-\theta)}{2} + \omega_o\right] \frac{\sin \frac{N(\omega-\theta)}{2}}{\sin \frac{\omega-\theta}{2}} \qquad \text{(Eq. A-10)}$$

Similarly the second term is

$$\sum_{t=1}^{N} A \cos(\omega t+\omega_o) \sin \theta t = \frac{A}{2} \sum_{t=1}^{N} \sin\left[(\omega+\theta)t+\omega_o\right]$$

$$- \frac{A}{2} \sum_{t=1}^{N} \sin\left[(\omega-\theta)t+\omega_o\right]$$

$$= \frac{A}{2} \sin \left[\frac{(N+1)(\omega+\theta)}{2} + \omega_o\right] \frac{\sin \frac{N(\omega+\theta)}{2}}{\sin \frac{\omega+\theta}{2}}$$

$$- \frac{A}{2} \sin \left[\frac{(N+1)(\omega-\theta)}{2} + \omega_o\right] \frac{\sin \frac{N(\omega-\theta)}{2}}{\sin \frac{\omega-\theta}{2}} \qquad \text{(Eq. A-11)}$$

Let

$$c_+ = \cos \left[\frac{(N+1)(\omega+\theta)}{2} + \omega_o\right] \qquad s_+ = \sin \left[\frac{(N+1)(\omega+\theta)}{2} + \omega_o\right]$$

$$c_- = \cos \left[\frac{(N+1)(\omega-\theta)}{2} + \omega_o\right] \qquad s_- = \sin \left[\frac{(N+1)(\omega-\theta)}{2} + \omega_o\right]$$

$$k_+ = \frac{\sin \frac{N(\omega+\theta)}{2}}{\sin \frac{\omega+\theta}{2}} \qquad \text{(Eq. A-12)}$$

$$k_- = \frac{\sin \frac{N(\omega-\theta)}{2}}{\sin \frac{\omega-\theta}{2}}$$

Now Eq. A-2 can be written, from Eq. A-10, 11, and 12, as

$$f_d(\theta) = \frac{1}{\pi N} \frac{A^2}{4} \left[(c_+k_+ + c_-k_-)^2 + (s_+k_+ - s_-k_-)^2\right]$$

$$= \frac{A^2}{4\pi N} \left[(c_+^2 + s_+^2)k_+^2 + 2(c_+c_- - s_+s_-)k_+k_- + (c_-^2 + s_-^2)k_-^2\right] \qquad \text{(Eq. A-13)}$$

but

$$c_+^2 + s_+^2 = 1 \qquad \text{(Eq. A-14)}$$

$$c_-^2 + s_-^2 = 1 \qquad \text{(Eq. A-15)}$$

and

$$c_+c_- - s_+s_- = \cos\left[\frac{(N+1)(\omega+\theta)}{2} + \omega_o + \frac{(N+1)(\omega-\theta)}{2} + \omega_o\right]$$

$$= \cos\left[(N+1)\omega + 2\,\omega_o\right] \qquad \text{(Eq. A-16)}$$

Thus Eq. A-13 can be written as

$$f_d(\theta) = \frac{A^2}{4\pi N}\left\{\frac{\sin^2 \frac{N(\omega+\theta)}{2}}{\sin^2 \frac{\omega+\theta}{2}} + 2\cos\left[(N+1)\,\omega + 2\,\omega_o\right]\right.$$

$$\left.\frac{\sin\frac{N(\omega+\theta)}{2}\,\sin\frac{N(\omega-\theta)}{2}}{\sin\frac{\omega+\theta}{2}\quad\sin\frac{\omega-\theta}{2}} + \frac{\sin^2\frac{N(\omega-\theta)}{2}}{\sin^2\frac{\omega-\theta}{2}}\right\} \qquad \text{(Eq. A-17)}$$

APPENDIX B: SPECTRUM OF WHITE NOISE

If a_t is an independently and identically distributed random variable of mean 0 and variance σ_a^2 then the spectrum of a_t, denoted by $f_a(\theta)$, is

$$f_a(\theta) = \frac{\sigma_a^2}{\pi} \qquad 0 \le \theta \le \pi \tag{Eq. B-1}$$

Proof

$$f_a(\theta) = \frac{1}{\pi N} \left| \sum_{t=1}^{N} a_t e^{i\theta t} \right|^2$$

$$= \frac{1}{\pi N} \left[\left(\sum_{t=1}^{N} a_t e^{i\theta t} \right) \left(\sum_{t=1}^{N} a_t e^{-i\theta t} \right) \right]$$

$$= \frac{1}{\pi N} \sum_t \sum_k a_t a_k e^{i\theta(t-k)} \tag{Eq. B-2}$$

let $\ell = t-k$, therefore $k = t-\ell$ and

$$f_a(\theta) = \frac{1}{\pi N} \sum_\ell \left(\sum_t a_t a_{t-\ell} \right) e^{i\theta\ell} \tag{Eq. B-3}$$

but by hypothesis

$$\sum_t a_t a_{t-\ell} = 0 \quad \ell \neq 0 \tag{Eq. B-4}$$

Hence

$$f_a(\theta) = \frac{1}{\pi N} \sum_{t=1}^{N} a_t^2$$

$\therefore$

$$f_a(\theta) = \frac{\sigma_a^2}{\pi} \tag{Eq. B-5}$$

APPENDIX C: GREEN'S FUNCTION

Green's function is defined such that

$$X_t = \sum_{j=0}^{\infty} G_j a_{t-j} \qquad \text{(Eq. C-1)}$$

Therefore an ARMA(2,2) model can be written as

$$\phi(B)X_t = \theta(B)a_t$$

$$(1-\phi_1 B-\phi_2 B^2)X_t = (1-\theta_1 B-\theta_2 B^2)a_t$$

$$(1-\phi_1 B-\phi_2 B^2)(1+G_1 B+G_2 B^2+\ldots)a_t = (1-\theta_1 B-\theta_2 B^2)a_t \qquad \text{(Eq. C-2)}$$

Equating like powers of B gives an implicit (recursive) form of Green's function.

$$G_0 = 1 \qquad : \qquad G_o = 1$$

$$G_1 - \phi_1 = -\theta_1 \qquad : \qquad G_1 = \phi_1 - \theta_1$$

$$G_2 - \phi_1 G_1 - \phi_2 = -\theta_2 \qquad : \qquad G_2 = \phi_1 G_1 + \phi_2 - \theta_2$$

$$G_3 - \phi_1 G_2 - \phi_2 G_1 = 0 \qquad : \qquad G_3 = \phi_1 G_2 + \phi_2 G_1$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$G_j - \phi_1 G_{j-1} - \phi_2 G_{j-2} = 0 \quad : \quad G_j = \phi_1 G_{j-1} + \phi_2 G_{j-2} \qquad \text{(Eq. C-3)}$$

for $j = 3,4,\ldots$

Green's function can also be written in an explicit (analytic) form by using the implicit form as the initial conditions and the general solution to a second order difference equation. Assuming distinct roots

$$G_j = g_1\lambda_1^j + g_2\lambda_2^j \qquad j = 1,2,\ldots \qquad \text{(Eq. C-4)}$$

Therefore

$$G_1 = g_1\lambda_1 + g_2\lambda_2 = \phi_1 - \theta_1 \qquad \text{(Eq. C-5)}$$

$$= \lambda_1 + \lambda_2 - \theta_1$$

and

$$G_2 = g_1\lambda_1^2 + g_2\lambda_2^2 = \phi_1^2 - \phi_1\theta_1 + \phi_2 - \theta_2 \qquad \text{(Eq. C-6)}$$

$$= (\lambda_1 + \lambda_2)^2 - (\lambda_1 + \lambda_2)\theta_1 + (-\lambda_1\lambda_2) - \theta_2$$

From Eq. C-5

$$g_2 = \frac{\phi_1 - \theta_1 - g_1\lambda_1}{\lambda_2} \qquad \text{(Eq. C-7)}$$

Substituting Eq. C-7 into Eq. C-6 gives

$$g_1\lambda_1^2 + \frac{\phi_1 - \theta_1 - g_1\lambda_1}{\lambda_2}\lambda_2^2 = \phi_1^2 - \phi_1\theta_1 + \phi_2 - \theta_2 \qquad \text{(Eq. C-8)}$$

$$g_1(\lambda_1^2 - \lambda_1\lambda_2) = \phi_1^2 - \phi_1\theta_1 + \phi_2 - \theta_2 - \phi_1\lambda_2 + \theta_1\lambda_2 \qquad \text{(Eq. C-9)}$$

$$g_1 = \frac{\lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 - \lambda_1\theta_1 - \lambda_2\theta_1 - \lambda_1\lambda_2 - \theta_2 - \lambda_1\lambda_2 - \lambda_2^2 + \theta_1\lambda_2}{\lambda_1(\lambda_1 - \lambda_2)}$$

$$g_1 = \frac{\lambda_1(\lambda_1 - \theta_1) - \theta_2}{\lambda_1(\lambda_1 - \lambda_2)} \qquad \text{(Eq. C-10)}$$

Substituting Eq. C-10 into Eq. C-7 gives

$$g_2 = \frac{\lambda_1 + \lambda_2 - \theta_1 - \dfrac{\lambda_1(\lambda_1 - \theta_1) - \theta_2}{\lambda_1 - \lambda_2}}{\lambda_2}$$

$$= \frac{(\lambda_1 + \lambda_2)(\lambda_2 - \lambda_1) - \theta_1(\lambda_2 - \lambda_1) + \lambda_1(\lambda_1 - \theta_1) - \theta_2}{\lambda_2(\lambda_2 - \lambda_1)}$$

$$= \frac{\lambda_1\lambda_2 + \lambda_2^2 - \lambda_1^2 - \lambda_1\lambda_2 - \theta_1\lambda_2 + \theta_1\lambda_1 + \lambda_1^2 - \lambda_1\theta_1 - \theta_2}{\lambda_2(\lambda_2 - \lambda_1)}$$

$$g_2 = \frac{\lambda_2(\lambda_2 - \theta_1) - \theta_2}{\lambda_2(\lambda_2 - \lambda_1)} \qquad \text{(Eq. C-11)}$$

Now G_0 must equal one so consider

$$g_1 + g_2 = \frac{\lambda_1(\lambda_1 - \theta_1) - \theta_2}{\lambda_1(\lambda_1 - \lambda_2)} + \frac{\lambda_2(\lambda_2 - \theta_1) - \theta_2}{\lambda_2(\lambda_2 - \lambda_1)}$$

$$= \frac{\lambda_1^2\lambda_2 - \lambda_1\lambda_2\theta_1 - \lambda_2\theta_2 - \lambda_1\lambda_2^2 + \lambda_1\lambda_2\theta_1 + \lambda_1\theta_2}{\lambda_1\lambda_2(\lambda_1 - \lambda_2)}$$

$$= \frac{\lambda_1\lambda_2 + \theta_2}{\lambda_1\lambda_2}$$

$\therefore$

$$g_1 + g_2 = 1 - \frac{\theta_2}{\phi_2} \qquad \text{(Eq. C-12)}$$

In summary, then, the explicit form of Green's function is

$$G_0 = \frac{\theta_2}{\phi_2} + g_1 + g_2 = 1$$

$$G_j = g_1\,\lambda_1^j + g_2\,\lambda_2^j \qquad j = 1,2,\ldots \qquad \text{(Eq. C-13)}$$

where

$$g_1 = \frac{\lambda_1(\lambda_1 - \theta_1) - \theta_2}{\lambda_1(\lambda_1 - \lambda_2)}$$

and

$$g_2 = \frac{\lambda_2(\lambda_2 - \theta_1) - \theta_2}{\lambda_2(\lambda_2 - \lambda_1)}$$

The explicit form of Green's function can also be written in a trigonometric form if λ and ν are complex conjugates.

Let

$$\lambda_1 = re^{i\rho}$$

$$\lambda_2 = re^{-i\rho} \qquad \text{(Eq. C-14)}$$

$$\nu_1 = se^{i\sigma}$$

$$\nu_2 = se^{-i\sigma}$$

Then

$$G_j = \frac{\lambda_1(\lambda_1 - \theta_1) - \theta_2}{\lambda_1(\lambda_1 - \lambda_2)} (re^{i\rho})^j + \frac{\lambda_2(\lambda_2 - \theta_1) - \theta_2}{\lambda_2(\lambda_2 - \lambda_1)} (re^{-i\rho})^j$$

$$= \frac{r^j}{\lambda_1 - \lambda_2} \left[\left(\lambda_1 - \theta_1 - \frac{\theta_2}{\lambda_1}\right) e^{ij\rho} - \left(\lambda_2 - \theta_1 - \frac{\theta_2}{\lambda_2}\right) e^{-ij\rho}\right]$$

$$= \frac{r^j}{\lambda_1 - \lambda_2} \left[\left(\lambda_1 - \theta_1 - \frac{\theta_2}{\lambda_1}\right) (\cos j\rho + i \sin j\rho)\right.$$

$$\left. - \left(\lambda_2 - \theta_1 - \frac{\theta_2}{\lambda_2}\right) (\cos j\rho - i \sin j\rho)\right]$$

$$= \frac{r^j}{\lambda_1 - \lambda_2} \left[\left(\lambda_1 - \lambda_2 - \frac{\theta_2}{\lambda_1} + \frac{\theta_2}{\lambda_2}\right) \cos j\rho\right.$$

$$\left. + i\left(\lambda_1 + \lambda_2 - 2\theta_1 - \frac{\theta_2}{\lambda_1} - \frac{\theta_2}{\lambda_2}\right) \sin j\rho\right]$$

$$= r^j \left[\frac{\lambda_1\lambda_2 + \theta_2}{\lambda_1\lambda_2} \cos\rho j + i \frac{\phi_1 - 2\theta_1 + \frac{\phi_1\theta_2}{\phi_2}}{\lambda_1 - \lambda_2} \sin\rho j \right]$$

$$= r^j \left[\left(1 - \frac{\theta_2}{\phi_2}\right) \cos\rho j + \frac{\left(\phi_1 - 2\theta_1 + \frac{\phi_1\theta_2}{\phi_2}\right)}{\sqrt{-(\phi_1^2 + 4\,\phi_2)}} \sin\rho j \right] \qquad \text{(Eq. C-15)}$$

Let

$$B = \frac{\phi_1 - 2\theta_1 + \frac{\phi_1\theta_2}{\phi_2}}{\sqrt{-(\phi_1^2 + 4\,\phi_2)}} \qquad \text{(Eq. C-16)}$$

Thus

$$G_j = r^j \left[\left(1 - \frac{\theta_2}{\phi_2}\right) \cos\rho j + B \sin\rho j \right] \qquad \text{(Eq. C-17)}$$

Define

$$R = \sqrt{\left(1 - \frac{\theta_2}{\phi_2}\right)^2 + B^2}$$

and (Eq. C-18)

$$\beta = \tan^{-1} \left[\frac{B}{1 - \frac{\theta_2}{\phi_2}} \right]$$

per Figure C-1.

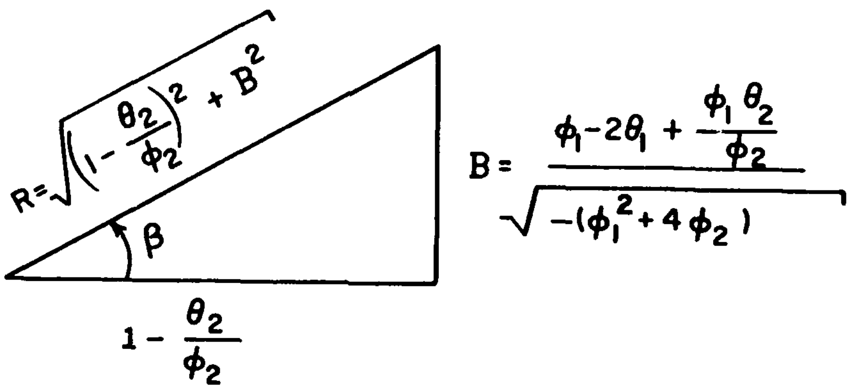

Figure C-1. Illustration of β

Therefore,

$$G_j = r^j \, R \left[\cos\beta \, \cos\rho j + \sin\beta \, \sin\rho j\right] \qquad \text{(Eq. C-19)}$$

Finally

$$G_j = r^j \, R \cos\,(\rho j - \beta) \qquad j = 1,2,\ldots \qquad \text{(Eq. C-20)}$$

where

$$\lambda = re^{i\rho}$$

$$R = \sqrt{\left(1 - \frac{\theta_2}{\phi_2}\right)^2 + B^2}$$

$$B = \frac{\phi_1 - 2\theta_1 + \dfrac{\phi_1\theta_2}{\phi_2}}{\sqrt{-(\phi_1^2 + 4\,\phi_2)}}$$

$$\beta = \tan^{-1}\left[\frac{B}{1 - \dfrac{\theta_2}{\phi_2}}\right]$$

We now derive an expression for

$$\sum_{j=0}^{\infty} G_j^2$$

$$\sum_{j=0}^{\infty} G_j^2 = 1 + \sum_{j=1}^{\infty} \left[r^j \ R \cos \ (\rho j - \beta)\right]^2$$

$$= 1 + R^2 \sum_{j=1}^{\infty} r^{2j} \cos^2 (\rho j - \beta)$$

$$= 1 + R^2 \sum_{j=1}^{\infty} r^{2j} \left[\frac{e^{i(\rho j - \beta)}}{2} + \frac{e^{-i(\rho j - \beta)}}{2}\right]^2$$

$$= 1 + \frac{R^2}{4}\left[\sum_{j=1}^{\infty} r^{2j} \ e^{i2(\rho j - \beta)} + \sum_{j=1}^{\infty} 2r^{2j} + \sum_{j=1}^{\infty} r^{2j} \ e^{-i2(\rho j - \beta)}\right]$$

(Eq. C-21)

Consider the first term of Eq. C-21

$$\sum_{j=1}^{\infty} r^{2j} \ e^{i2(\rho j - \beta)} = e^{-i2\beta} \frac{r^2 \ e^{i2\rho}}{1 - r^2 \ e^{i2\rho}} \qquad \text{(Eq. C-22)}$$

Similarly the third term is

$$\sum_{j=1}^{\infty} r^{2j} \ e^{-i2(\rho j - \beta)} = e^{i2\beta} \frac{r^2 \ e^{-i2\rho}}{1 - r^2 \ e^{-i2\rho}} \qquad \text{(Eq. C-23)}$$

Next, the second term is

$$\sum_{j=1}^{\infty} 2r^{2j} = 2 \frac{r^2}{1 - r^2} \qquad \text{(Eq. C-24)}$$

Thus, Eq. C-21 can be written as

$$\sum_{j=0}^{\infty} G_j^2 = 1 + \frac{R^2}{4}\left[e^{-i2\beta}\frac{r^2\, e^{i2\rho}}{1-r^2\, e^{i2\rho}} + \frac{2r^2}{1-r^2} + e^{i2\beta}\frac{r^2\, e^{-i2\rho}}{1-r^2\, e^{-i2\rho}}\right]$$

$$= 1 + \frac{R^2 r^2}{4}\left[\frac{e^{i(2\rho-2\beta)} - r^2 e^{-i2\beta} + e^{-i(2\rho-2\beta)} - r^2 e^{i2\beta}}{1 - r^2(e^{i2\rho} + e^{-i2\rho}) + r^4} + \frac{2}{1-r^2}\right]$$

$$\sum_{j=0}^{\infty} G_j^2 = 1 + \frac{R^2 r^2}{2}\left[\frac{\cos\,(2(\rho-\beta)) - r^2\cos 2\beta}{1 + r^4 - 2r^2\cos 2\rho} + \frac{1}{1-r^2}\right] \qquad \text{(Eq. C-25)}$$

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